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On zero-sum Ramsey numbers—stars

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Abstract

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Let $n \geqslant k \geqslant 2$ be positive integers, $k \mid n$. Let $\mathbb{Z}_k$ be the cyclic group of order k. Denote by $R(K_{1,n}, \mathbb{Z}_k)$ the minimal integer t such that for every $\mathbb{Z}_k$-coloring of the edges of K_t, (i.e., a function $c: E(K_t) \to \mathbb{Z}_k$), there is in K_t a copy of $K_{1,n}$ with the property that $\sum_{e \in E(K_{1,n})} c(e) \equiv 0 \pmod k$. Answering a problem raised by Bialostocki and Dierker we prove that if $k \mid n$ then

$$R(K_{1,n}, \mathbb{Z}_k) = \begin{cases} n+k-1 & n \equiv k \equiv 0 \pmod 2, \\ n+k & \text{otherwise.} \end{cases}$$

Some variants are also considered.

1. Introduction

Bialostocki and Dierker [1–4] raised the following interesting variant of the classical Ramsey Theorem: Let G be a graph having m edges and let $k \geqslant 2$ be an integer such that $k \mid m$, and let $\mathbb{Z}_k$ be the cyclic group of order k. Define $R(G, \mathbb{Z}_k)$ to be the minimal integer t such that for every $\mathbb{Z}_k$-coloring of the edges of K_t, i.e., a function $c: E(K_t) \to \mathbb{Z}_k$, there is in K_t a copy of G with the property that

$$\sum_{e \in E(G)} c(e) \equiv 0 \pmod k.$$

They proved in particular that

$$R(K_{1,n}, \mathbb{Z}_n) = \begin{cases} 2n & n \text{ is odd,} \\ 2n-1 & n \text{ is even} \end{cases}$$

and raised the problem of determining $R(K_{1,n}, \mathbb{Z}_k)$ for every k and n such that $k \mid n$.

Solving this problem is the main concern of this paper. Several preliminary results are needed in order to be able to deal with this problem. Some of them come from combinatorial number theory, e.g., the Erdős–Ginzburg–Ziv Theorem [6], and some are of graph theoretical nature, e.g., the decomposition of complete graphs into hamiltonian cycles.

The next section concerns these preliminaries. We shall then consider the determination of $R(K_{1,n}, \mathbb{Z}_k)$ as well as some variants.

Our notation is standard and we follow mainly [7]. Information on the Ramsey numbers for stars can be found in [5].

2. Preliminaries

The following general form of the Erdős–Ginzburg–Ziv Theorem [6], proved in 1961, is well known.

Theorem A [6]. *Let $\{a_1, a_2, \ldots, a_{(m+1)k-1}\}$ be a collection of integers. There exists a subset $I \subset \{1, 2, \ldots, (m+1)k-1\}$, $|I| = mk$ such that $\sum_{i\in I} a_i \equiv 0 \pmod{k}$.*

(The case $m = 1$ is the case usually cited in the literature.) Characterizing the extremal cases in the EGZ-Theorem, Bialostocki and Dierker [1] proved the following.

Theorem B [1]. *Let m be an integer, $m \geq 2$. If $a_1, a_2, \ldots, a_{2m-2}$ is a sequence of $2m-2$ residues modulo m and there are no m indices $1 \leq i_1 < i_2 < \cdots < i_m \leq 2m-2$ such that $a_{i_1} + a_{i_2} + \cdots + a_{i_m} \equiv 0 \pmod{m}$, then there are only two residue classes modulo m, such that $m-1$ of the a_i's belong to one of the classes and the remaining $m-1$ a_i's belong to the other class.*

We need the following extension of Theorem B.

Theorem 1. *Let $A = \{a_1, a_2, \ldots, a_{(m+1)k-2}\}$ be a collection of integers. Suppose there exists no subset $I \subset \{1, 2, \ldots, (m+1)k-2\}$, $|I| = mk$, such that $\sum_{i\in I} a_i \equiv 0 \pmod{k}$, then*

(1) *the members of A belong to exactly two residue classes of $\mathbb{Z}_k$ and further, each of these residue classes contains $-1 \pmod{k}$ members from A,*

(2) *if k is even, then the residue classes are of distinct parity.*

Proof. We prove the assertions of the theorem by induction on m. First we prove assertion (1). For $m = 1$ this is exactly the claim of Theorem B. Observe also that for $k = 2$ the theorem holds because of Theorem A. So we may assume $k \geq 3$. Suppose $A = \{a_1, \ldots, a_{(m+1)k-2}\}$ contains members from three residue classes $\bmod k$, say w.l.o.g., $a_{(m+1)k-4}$, $a_{(m+1)k-3}$, $a_{(m+1)k-2}$. Consider the remaining members, there are $(m+1)k-5 = mk+k-5$ members.

(i) If $k \geqslant 4$ then $mk + k - 5 \geqslant mk - 1$ and by Theorem A there is a subset I, $|I| = (m-1)k$, such that $\sum_{i \in I} a_i \equiv 0 \pmod{k}$.

We still have $2k - 2$ members of A which were not chosen, and also among them are the three last members of A, which belong to three residue classes. Hence by Theorem B we can find k members, not yet chosen, whose sum is 0 (mod k) resulting in a subset I_1, $|I_1| = mk$, $\sum_{i \in I_1} a_i \equiv 0 \pmod{k}$, contradicting the assumption of the theorem.

(ii) If $k = 3$ then of course $a_{(m+1)k-4} + a_{(m+1)k-3} + a_{(m+1)k-2} \equiv 0 \pmod{3}$, and there remain $mk - 2$ members which must belong to exactly two residue classes for otherwise by induction hypothesis there will be $(m-1)k$ members whose sum is 0 (mod 3) which together with the last three members of A give a collection of mk members whose sum is 0 (mod 3) contradicting the assumption.

If $m = 2$ there are just three possibilities to check, namely

$$\{0,0,1,1,1,2,0\} \Rightarrow 0+0+1+1+1+0 \equiv 0 \pmod{3},$$

$$\{0,0,2,2,1,2,0\} \Rightarrow 0+0+2+2+2+0 \equiv 0 \pmod{3},$$

$$\{1,1,2,2,1,2,0\} \Rightarrow 1+1+2+2+1+2 \equiv 0 \pmod{3}.$$

Hence assume $m \geqslant 3$, $mk - 2 \geqslant 7$, and we conclude that one of the residue classes contains at least three members whose sum is clearly 0 (mod 3), say the members are a_1, a_2, a_3. Now $B = A \backslash \{a_1, a_2, a_3\}$ contains $mk - 2$ members but of three residue classes becaus the last three members of A are in B. By the induction hypothesis, B contains $(m-1)k$ members whose sum is 0 (mod 3), adding to them a_1, a_2, a_3 we obtain a subcollection of mk members of A whose sum is 0 (mod 3), a contradiction. So we proved that there are exactly two residue classes in the extremal case. We have to show now that each residue class contains -1 (mod k) members from A.

Once again we use induction on m. For $m = 1$ this is the claim of Theorem B.

Assume $m \geqslant 2$. Clearly each of the residue classes contains at least $k - 1$ members for otherwise the other class would contain mk members whose sum is obviously 0 (mod k). Moreover, as $m \geqslant 2$, $(m+1)k - 2 \geqslant 3k - 2$, hence at least one of the residue classes contains at least k members, say w.l.o.g, a_{mk-1}, $a_{mk}, \ldots, a_{(m+1)k-2}$ whose sum is clearly 0 (mod k). Consider the remaining members $B = \{a_1, a_2, \ldots, a_{mk-2}\}$. By the induction hypothesis each of the residue classes must contain -1 (mod k) members of B, otherwise we would have mk members from A whose sum is 0 (mod k). Hence also in A each of the residue classes contains -1 (mod k) members. Lastly we have to show that if k is even, then the residue classes are of distinct parity. Suppose first that both residue classes are even, then all the members of A are even. Define $A' = \{a'_1, a'_2, \ldots, a'_{(m+1)k-2}\}$ by $a'_i = \frac{1}{2}a_i$.

Observe that $|A'| = (m+1)k - 2 \geqslant (2m+1)k/2 - 1$, hence by Theorem A there exists a subset $I \subset \{1, 2, \ldots, (m+1)k - 2\}$, $|I| = 2m \cdot k/2 = mk$, such that $\sum_{i \in I} a'_1 \equiv 0 \pmod{k/2}$, hence $\sum_{i \in I} a_i \equiv 0 \pmod{k}$.

Suppose now that both residue classes are odd. Define $A' = \{a'_1, a'_2, \ldots, a'_{(m+1)k-2}\}$ by $a'_i = a_i - 1$. Now the members of A' belong to exactly two even residue classes modulo k, and in these cases we infer that there exists a subset $I \in (1, 2, \ldots, (m+1)k-2\}$, $|I| = mk$, such that $\sum_{i\in I} a'_i \equiv 0 \pmod{k}$. But

$$\sum_{i\in I} a'_i = \sum_{i\in I} (a_i - 1) = \sum_{i\in I} a_i - mk;$$

hence also $\sum_{i\in I} a_i \equiv 0 \pmod{k}$ as needed. $\square$

We need one more result before stating our main result. This is a classical decomposition theorem and we refer the reader to the book of Harary [7].

Theorem C [7]. *Let K_n be the complete graph on n vertices.*

(1) *If $n \equiv 1 \pmod{2}$ then K_n is the edge-disjoint union of $(n-1)/2$ hamiltonian cycles C_n.*

(2) *If $n \equiv 0 \pmod{2}$ the K_n is the edge-disjoint union of $(n-1)/2$ hamiltonian cycles C_n and one perfect matching M.*

3. Main result

We start with the exact determination of the zero-sum Ramsey numbers for stars.

Theorem 2. *Let $n \geqslant k \geqslant 2$ be positive integers, and $k \mid n$ then*

$$R(K_{1,n}, \mathbb{Z}_k) = \begin{cases} n+k-1 & n \equiv k \equiv 0 \pmod{2}, \\ n+k & \textit{otherwise.} \end{cases}$$

Proof. We start by showing that

$$n+k-1 \leqslant R(K_{1,n}, \mathbb{Z}_k) \leqslant n+k.$$

For the lower bound consider the following construction: Take a copy of K_{n-1} and another copy of K_{k-1} and color all the edges in both graphs by the color 0. Color all the edges between them by color 1. One can check that there is no zero-sum copy (mod k) of $K_{1,n}$ in this construction; hence $R(K_{1,n}, \mathbb{Z}_k) \geqslant n+k-1$.

For the upper bound put $n = mk$ and consider a $\mathbb{Z}_k$-coloring of the edges of $K_{(m+1)k} = K_{n+k}$. The degree of each vertex in K_{n+k} is $(m+1)k-1$, hence by Theorem A there exists a subset of $n = mk$ edges incident with a vertex v whose sum is $0 \pmod{k}$ and we conclude that $R(K_{1,n}, \mathbb{Z}_k) \leqslant n+k$.

Now we have to consider three cases:

Case a: $n \equiv k \equiv 1 \pmod{2}$.

We have to improve the lower bound. Consider K_{n+k-1}, $n+k-1 \equiv 1 \pmod{2}$; hence by Theorem C(1), K_{n+k-1} is a union of hamiltonian cycles. Color $(n-1)/2$

hamiltonian cycles by the color 1, ($(n-1)/2$ an integer). Color $(k-1)/2$ hamiltonian cycles by the color 0, ($(k-1)/2$ an integer). In every vertex there are $n-1$ edges of color 1, $k-1$ edges of color 0, and there is no zero-sum $(\text{mod } k)$ copy of $K_{1,n}$.

Case b: $n \equiv 0 \pmod 2$, $k \equiv 1 \pmod 2$.

We have to improve the lower bound. Consider K_{n+k-1}, $n+k-1 \equiv 0 \pmod 2$; hence by Theorem C(2), K_{n+k-1} is a union of hamiltonian cycles and one perfect matching M. Color $(n-2)/2$ cycles and the perfect matching M by the color 1. Color $(k-1)/2$ cycles by the color 0. Again in every vertex there are $n-1$ edges of color 1, $k-1$ edges of color 0, and no zero-sum $(\text{mod } k)$ copy of $K_{1,n}$.

Case c: $n \equiv k \equiv 0 \pmod 2$.

We have to improve the upper bound. Consider K_{n+k-1}, $n+k-1 \equiv 1 \pmod 2$. The degree of every vertex is $n+k-2=(m+1)k-2$, $(n=mk)$. By Theorem 1 the only possibility to avoid a zero-sum $(\text{mod } k)$ copy of $K_{1,n}$ is when the edges incident with any vertex v are colored by two residue classes of distinct parity and each residue class contains $-1 \pmod k$ edges. But as k is even; each color class contains an odd number of edges incident with v. Recolor K_{n+k-1} by two colors according to the parity of the residue classes in the original $\mathbb{Z}_k$-coloring. Then in the new coloring c, each vertex is incident to an odd number of edges colored 0 and odd number of edges colored 1. The graph induced by the edges colored 0 has an odd number of vertices, namely $n+k-1$, and all the degrees are odd numbers, which is impossible. Hence, in this case $R(k_{1,n}, \mathbb{Z}_k) \leq n+k-1$. $\square$

One may now consider the following variant of Theorem 2. Call a star $K_{1,n}$ 'directed' if all the edges are either directed into the center of the star or out from the center. Denote by $R^*(K_{1,n}, \mathbb{Z}_k)$ the minimal integer t, such that for every $\mathbb{Z}_k$-coloring of the edges of K_t, and every orientation of its edges, there is a directed $K_{1,n}$ which is zero-sum modulo k.

Theorem 3. *Let $n \geq k \geq 2$ be positive integers, $k \mid n$. Then $R^*(K_{1,n}, \mathbb{Z}_k) = 2(n+k-1)$.*

Proof. Let us first establish the upper bound $R^*(K_{1,n}, \mathbb{Z}_k) \leq 2(n+k-1)$. Consider $K_{2(n+k-1)}$ with an arbitrary $\mathbb{Z}_k$-coloring and an arbitrary orientation. The degree of a vertex v is $2(n+k-1)-1$. Hence in each vertex v there must exist a directed $K_{1,n+k-1}$. By Theorem A this directed star must contain a zero-sum $(\text{mod } k)$ directed $K_{1,n}$.

For the lower bound consider $K_{2(n+k-1)-1}$. By Theorem C, $K_{2(n+k-1)-1}$ is the union of $n+k-2$ hamiltonian cycles. Color $n-1$ hamiltonian cycles by color 1 and orient them cyclically. Color $k-1$ hamiltonian cycles by color 0 and orient them cyclically. In each vertex v there are $2n-2$ edges colored 1, $n-1$ edges directed into v and $n-1$ edges directed from v. Also there are $2k-2$ edges colored 0, $k-1$ directed into v and $k-1$ directed from v. Hence there exists no zero-sum (mod k) directed $K_{1,n}$. □

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